Analysis and synthesis of randomly switched systems with known sojourn probabilities

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Analysis and Synthesis of Randomly Switched Systems with Known Sojourn Probabilities

Engang Tian, Dong Yue and Tai-cheng Yang

Abstract

In this paper, a new approach is proposed and investigated for the stability analysis and stabilizing controller design of randomly switched linear discrete systems. The approach is based on sojourn probabilities and it is assumed that these probabilities are known a prior. A sojourn probability for a subsystem is the probability of a switched system staying in that subsystem and can be obtained, for example from the history of the switched system. Two main theorems are proved in this paper. Theorem 1 gives a sufficient condition for a switched system with known sojourn probabilities to be mean square stable. Theorem 2 gives a sufficient condition for the design of a stabilizing controller. The applications of these theorems and the corresponding corollary and lemma are demonstrated by three numerical examples. A few directions/topics of future research are also proposed.

Keywords: Randomly switched systems, known sojourn probabilities, switching law, Lyapunov functional method

I. INTRODUCTION

Switched systems have wide applications in electronics, power systems, networked control systems, traffic control, etc. [1], [9], [15], [16]. Generally, switching actions can take place either passively or actively. For the former, an unpredictable sudden change in system dynamics/structure or an accidental activation of any subsystem can induce a switching activity. For the latter, switching may be introduced artificially to effectively model or control a system. Switching laws, in addition to control functions for subsystems, may be utilized to manipulate switched systems to achieve a better performance [16]. As a result, many complicated behavior/dynamics and fundamentally new properties have been demonstrated in switched systems. This makes the study of these systems more challenging and

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more interesting. There is an increasing research from the scientific community in switched systems, in particular for linear continuous and discrete systems - for example see a recent survey paper [9].

There are two categories of research. In the first category, switching between different subsystems can be designed to achieve the desired stability and/or performance. For example, there exists a class of unstable linear and nonlinear systems which can be stabilized by well-designed switching control schemes, but cannot be stabilized by any continuous static state feedback control. In the second category, one does not design switching laws, but concentrates on the stability analysis of the system and how to design stabilizing controls for subsystems. The study in this paper belongs to this category.

In the study of switched systems, first, switching between subsystems can be abrupt or arbitrarily. Section II of the survey paper [9] gives a good review of the stability conditions for arbitrarily switched systems. Secondly, in contrast to arbitrarily switching, there is a class of switched systems where an event to trigger a switching to another subsystem can be completely defined by the trajectory of the system state variables. Interested readers may refer to a brief overview presented in the introduction of a recent paper [11] and the references therein. Thirdly, the stability analysis can be based on the knowledge of the minimum time or average time of the system dynamics staying in subsystems. This leads to a dwell-time or average dwell-time based approach (see Section III of [9] for a survey). Due to uncertainties in system model and measurements, and more importantly due to the complexity of multi-conditions of a real-world event to trigger a switching, the fourth approach is also widely studied in recent years. Under this approach, a switched linear system is modeled as a Markovian Jump System (MJS) [2], [10], [26], [30]–[32], where the switching is governed by a Markov process. A completely known or a partially known Transition Probability Matrix (TPM) is used in an MJS to describe the probability of switching from one subsystem to another. Naturally, instead of asymptotical stability now the stability is defined as mean-square stability.

In this paper, we propose and investigate an alternative approach for the stability analysis and stabilizer design for switched linear discrete systems. This is based on the fact that over a long time horizon - regardless the statistics of switching actions from one subsystem to another such as modeled by a TPM, which is usually difficult to obtain - the probability of the system staying in a particular subsystem can be easily obtained, for example from the history of the system in various subsystems. For each subsystem, we call this probability *sojourn probability* and assume that this is known a prior. Our approach is to study switched systems based on Known Sojourn Probabilities (KSP). It appears that such an approach has not been reported in published literatures.

Within the KSP approach to catch the statistics of the duration of system dynamics remaining in different subsystems, another part of the system modeling is the dynamics of subsystems. In this paper, the study is based on a general time delay subsystem model. This class of switched systems, as shown in Section 2.1, can represent many models of problems currently studied by control professionals. Further study based on the KSP approach for other types of subsystem dynamics is our planned work in the future.

Our direct motivation for the proposed approach in this paper is the difficulties in the MJS approach to obtain the required TPM in applications. A typical example is applying the MJS approach to study the stability of a Networked Control Systems (NCS). Packet dropouts and channel communication delays are to be modeled by Markov Chains with a usual assumption that all the transition probabilities can be measured [14], [24], [27]. However, such statistics of transition probabilities is usually hard or costly to obtain [30]. The same problems may arise in other practical systems with jumps. On the other hand, sojourn probabilities in most cases are relatively easy to obtain. Furthermore, for a random process if the TPM is known, then the sojourn probabilities can be deduced from it, but not vise versa. The links and comparisons between the MJS and switched systems with KSP are one of the topics in our further research.

The rest of this paper is organized as follows: the modeling method of switched system with KSP is proposed in Section 2.1, it can be seen as a generalization of systems with many practical problems, as shown at the end of this subsection. By using a Lyapunov functional method, mean square stability criteria is obtained for switched systems with KSP, as shown in Theorem 1 of Subsection 2.2. The controller design method is proposed in Theorem 2 of Subsection 2.3. Three numerical examples are shown in Section 3 to demonstrate the usage and the advantages of the proposed approach. The paper ends with a short conclusion and a summary of five possible directions/topics for further research.

II. Stability analysis and stabilizing controller design based on the KSP

A. Problem formulation

Consider a discrete-time switched system with delays:

$$x(k+1) = A_{\sigma(k)}x(k) + A_{d\sigma(k)}x(k - d_{\sigma(k)}(k)) + B_{\sigma(k)}u(k) \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control input, $\sigma(k) : Z^+ = \{0, 1, 2, ...\} \rightarrow \{1, 2, ..., N\} \triangleq \Omega$ is the switching actions independent of the state. A_i, A_{di}, B_i $(i \in \Omega)$ are matrices with compatible dimensions for the i^{th} subsystem, $d_i(k)$ is the time-varying delay of the i^{th} subsystem satisfying $d_i^m < d_i(k) \leq d_i^M$. The controller is

$$u(k) = K_{\sigma(k)}x(k) \tag{2}$$

where $K_i (i \in \Omega)$ is the controller feedback gain of the i^{th} subsystem to be designed. Combining (2) and (1) leads to

$$x(k+1) = \left(A_{\sigma(k)} + B_{\sigma(k)}K_{\sigma(k)}x(k)\right)x(k) + A_{d\sigma(k)}x(k - d_{\sigma(k)}(k)) \tag{3}$$

In this paper, the probability of $\sigma(k)=i$ is assumed to be known, i.e.,

$$\Pr\{\sigma(k)=i\}=\alpha_i, \sum_{i=1}^{N}\alpha_i=1 \tag{4}$$

where $\alpha_i \in (0,1)$ is the sojourn probability of a switched system staying in the i^{th} subsystem. It is not difficult to obtain the statistic information α_i. In mathematical term

$$\alpha_i=\lim_{k\to\infty}\frac{k_i}{k}$$

where k_i is the times of $\sigma(k)=i$ in the interval $[1,k], k\in Z^+$.

A set of Bernoulli distributed processes $\gamma_i(k): Z^+ \to \{0,1\}$ is used in this paper:

$$\gamma_i(k)=\begin{cases} 1, & \sigma(k)=i \\ 0, & \sigma(k)\neq i \end{cases}, i\in\Omega, k\in Z^+, \tag{5}$$

and for any $k\in Z^+$

$$\sum_{i=1}^{N}\gamma_i(k)=1, \mathbb{E}\{\gamma_i(k)\}=\alpha_i, \sum_{i=1}^{N}\alpha_i=1. \tag{6}$$

The first equation in (6) is to guarantee that there is only one active subsystem at any time. Based on (3)-(6), the system model for the switched system is

$$x(k+1) = \sum_{i=1}^{N}\gamma_i(k)\left\{(A_i+B_iK_i)\,x(k)+A_{di}x(k-d_i(k))\right\} \tag{7}$$

$$x(k) = \phi(k), k=-d^M,-d^M+1,...,0 \tag{8}$$

where $d^M=\max\{d_i^M, i\in\Omega\}, \phi(k)$ is the initial state of $x(k)$.

Remark 1: The system model as shown in (7) is used by many researchers with some further assumptions. For example, in [5], [25], where the switching is assumed to be completely arbitrary; while in this paper, sojourn probabilities are assumed known a prior.

The purpose of this paper is to study the mean square stability of the randomly switched system (7) with KSP and to study stabilizing controller design.

Definition 1: The system defined by (7) with switching law (5) is mean-square stable if there exists a scalar $c>0$ such that

$$\mathbb{E}\left\{\sum_{k=0}^{\infty}\|x(k)\|^2\right\}\leq c\sup_{-d^M\leq i\leq 0}\mathbb{E}\left\{\|\phi(i)\|\right\}^2$$

Before the stability analysis for a switched system of (7) in the next subsection, it is worth pointing out that the system studied in this paper is a "switched model" of many systems, for which the original formulations of the systems are not in a form of switched systems; and that the results of this paper can be applied to the stability analysis and controller design of these systems if the statistic information - sojourn probabilities - is known.

First consider some systems with single or multiple missing measurements [3], [18], [20], [22], [23], or having sensor failure [8], or packet loss in communications [21], [24]. The system dynamics can be modeled as switching between a "normal-functioned" subsystem (without measurement missing, sensor failure or packet loss) and an (or more) "abnormal-functioned" subsystem(s); and the probability of staying in a particular subsystem is known.

Next consider systems with three different types of random delay.

1) For the communication delay in NCS less than two sampling intervals, the NCS is modeled as switched systems with two subsystems in [27]: subsystem with one step delay and subsystem without delay.

2) For a system having large communication delays, the range of the delay is divided into 2 subintervals in [17], [28], [29]. The system therefore can be modeled as a switched system with two subsystems. The sojourn probability of each subsystem is the probability of delay falling into each subinterval.

3) When the probabilities of time delay taking all the values are known, the system is modeled as a switched system with m subsystems (m is the number of delay values) [7]. Similarly, the probability of delay taking each value is the sojourn probability of each subsystem.

Switched systems with KSP information (7) can be seen as a generalization of the proposed system models in [3], [7], [8], [17], [18], [20]–[24], [27]–[29].

B. Stability Analysis for Randomly Switched Systems

In this section, first a stability theorem for an unforced system

$$x(k+1) = \sum_{i=1}^{N} \gamma_i(k) \left\{ A_i x(k) + A_{di} x(k - d_i(k)) \right\} \tag{9}$$

is presented and proved.

Theorem 1: System (9) with switching law (5) is mean square stable if there exist matrices $P > 0, T_i > 0, Q_i > 0, S_i > 0$ and $R_i > 0 (i \in \Omega)$ with compatible dimensions such that

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} \\ * & \Xi_{22} & 0 & 0 \\ * & * & \Xi_{33} & 0 \\ * & * & * & \Xi_{44} \end{bmatrix} < 0 \tag{10}$$

where $d_i^r = d_i^M - d_i^m (i \in \Omega)$,

$$\Xi_{11} = \begin{bmatrix} M_{11} & \mathscr{X}_1 \\ \mathscr{X}_1^T & \mathscr{X}_2 \end{bmatrix}$$

$$
\begin{aligned}
\mathscr{X}_1 &= \left[\begin{array}{ccc} M_{12} & \cdots & M_{1N} \end{array}\right], \mathscr{X}_2 = diag\{M_{22}, \cdots, M_{NN}\}, \\
M_{11} &= -P + \sum_{i=1}^{N} T_i - \sum_{i=1}^{N} S_i, M_{1l} = \left[\begin{array}{ccc} S_l & 0 & 0 \end{array}\right], l \in \Omega/\{1\}, \\
M_{ll} &= \left[\begin{array}{ccc} -S_l - R_l + Q_l - T_l & R_l & 0 \\ R_l & -2R_l & R_l \\ 0 & R_l & -R_l - Q_l \end{array}\right], l \in \Omega/\{1\}, \\
\Xi_{12} &= \left[\begin{array}{ccc} \Lambda_1 & \cdots & \Lambda_N \end{array}\right], \Xi_{22} = diag\{-P, -P, \cdots, -P\}, \\
\Lambda_i &= \begin{array}{c} \left[\begin{array}{cccccccccccc} \sqrt{\alpha_i} P A_i & 0 & 0 & 0 & \cdots & 0 & \sqrt{\alpha_i} P A_{di} & 0 & \cdots & 0 & 0 & 0 \end{array}\right]^T \\ \begin{array}{cccccccccccc} \quad 1 \quad & & & & & & \qquad\qquad\qquad 3i & & & & & \end{array} \end{array}, \\
\Xi_{13} &= \left[\begin{array}{ccc} d_1^m \Upsilon S_1 & \cdots & d_N^m \Upsilon S_N \end{array}\right], \Xi_{14} = \left[\begin{array}{ccc} d_1^r \Upsilon R_1 & \cdots & d_N^r \Upsilon R_N \end{array}\right], \\
\Xi_{33} &= diag\{-S_1, \cdots, -S_1, \cdots, -S_N, \cdots, -S_N\}, \\
\Xi_{44} &= diag\{-R_1, \cdots, -R_1, \cdots, -R_N, \cdots, -R_N\}, \\
\Upsilon &= \left[\begin{array}{ccc} \Upsilon_1 & \cdots & \Upsilon_N \end{array}\right], \\
\Upsilon_i &= \begin{array}{c} \left[\begin{array}{cccccccccccc} \sqrt{\alpha_i}\left(A_i - I\right) & 0 & 0 & 0 & \cdots & 0 & \sqrt{\alpha_i} A_{di} & 0 & \cdots & 0 & 0 & 0 \end{array}\right]^T \\ \begin{array}{cccccccccccc} \quad 1 \quad & & & & & & \qquad\qquad\qquad 3i & & & & & \end{array} \end{array}.
\end{aligned}
$$

Proof: Consider the unforced system (9) and construct a Lyapunov functional

$$
\begin{aligned}
V(k) &= \sum_{i=1}^{3} V_i(k) \qquad (11) \\
V_1(k) &= x^T(k) P x(k) \\
V_2(k) &= \sum_{i=1}^{N} \left(\sum_{l=k-d_i^m}^{k-1} x^T(l) T_i x(l) + \sum_{l=k-d_i^M}^{k-d_i^m-1} x^T(l) Q_i x(l) \right) \\
V_3(k) &= \sum_{i=1}^{N} \left(d_i^m \sum_{l=-d_i^m}^{-1} \sum_{\lambda=k+l}^{k-1} y^T(\lambda) S_i y(\lambda) + d_i^r \sum_{l=-d_i^M}^{-d_i^m-1} \sum_{\lambda=k+l}^{k-1} y^T(\lambda) R_i y(\lambda) \right)
\end{aligned}
$$

where $i \in \Omega, d_i^r = d_i^M - d_i^m, y(k) = x(k+1) - x(k)$. Denote $\mathbb{E}\{\Delta V(k)\} = \mathbb{E}\{V(k+1) - V(k)\}$ and notice that

$$
\mathbb{E}\{\gamma_i(k)\gamma_j(k)\} = \begin{cases} \mathbb{E}\{\gamma_i^2(k)\} = \alpha_i, i = j, \\ 0, i \neq j. \end{cases} \tag{12}
$$

Therefore

$$
\begin{aligned}
\mathbb{E}\{\Delta V_1(k)\} &= \mathbb{E}\left\{ \sum_{i=1}^{N} \alpha_i \left[A_i x(k) + A_{di} x\left(k - d_i(k)\right)\right]^T P \left[A_i x(k) + A_{di} x\left(k - d_i(k)\right)\right] - x^T(k) P x(k) \right\} \\
\mathbb{E}\{\Delta V_2(k)\} &= \mathbb{E}\left\{ \sum_{i=1}^{N} \left[x^T(k) T_i x(k) + x^T(k - d_i^m)\left(Q_i - T_i\right) x(k - d_i^m) - x^T(k - d_i^M) Q_i x(k - d_i^M)\right] \right\}
\end{aligned}
$$

$$\begin{aligned}\mathbb{E}\left\{\Delta V_3(k)\right\} &= \mathbb{E}\left\{\sum_{i=1}^{N}\left\{\left(d_i^m\right)^2 y^T(k)S_iy(k)+\left(d_i^r\right)^2 y^T(k)R_iy(k)\right.\right.\\ &\left.\left. -d_i^m\sum_{l=k-d_i^m}^{k-1}y^T(l)S_iy(l)-d_i^r\sum_{l=k-d_i^M}^{k-d_i^m-1}y^T(l)R_iy(l)\right\}\right\}\end{aligned} \tag{13}$$

where

$$-d_i^m\sum_{l=k-d_i^m}^{k-1}y^T(l)S_iy(l) \leq \begin{bmatrix} x(k)\\ x(k-d_i^m)\end{bmatrix}^T\begin{bmatrix}-S_i & S_i\\ S_i & -S_i\end{bmatrix}\begin{bmatrix} x(k)\\ x(k-d_i^m)\end{bmatrix} \tag{14}$$

$$-d_i^r\sum_{l=k-d_i^M}^{k-d_i^m-1}y^T(l)R_iy(l) \leq \begin{bmatrix} x(k-d_i^m)\\ x(k-d_i(k))\\ x(k-d_i^M)\end{bmatrix}^T\begin{bmatrix}-R_i & R_i & 0\\ R_i & -2R_i & R_i\\ 0 & R_i & -R_i\end{bmatrix}\begin{bmatrix} x(k)\\ x(k-d_i(k))\\ x(k-d_i^M)\end{bmatrix} \tag{15}$$

Combining (11)-(15)

$$\begin{aligned}\mathbb{E}\left\{\Delta V(k)\right\} &\leq \mathbb{E}\left\{\xi^T(k)\Xi_{11}\xi(k)+\sum_{i=1}^{N}\left[\left(d_i^m\right)^2 y^T(k)S_iy(k)+\left(d_i^r\right)^2 y^T(k)R_iy(k)\right]\right.\\ &\left.+\sum_{i=1}^{N}\alpha_i\left[A_ix(k)+A_{di}x\left(k-d_i(k)\right)\right]^T P\left[A_ix(k)+A_{di}x\left(k-d_i(k)\right)\right]\right\}\end{aligned} \tag{16}$$

By using Schur complement, it can be concluded from (10) and (16) that

$$\mathbb{E}\left\{\Delta V(k)\right\} \leq \mu\mathbb{E}\left\{x^T(k)x(k)\right\} \tag{17}$$

Similar to the analysis in [17], one can conclude that

$$\mathbb{E}\left\{\sum_{k=0}^{\infty}\|x(k)\|^2\right\} \leq c\sup_{-d^M\leq i\leq 0}\mathbb{E}\left\{\|\phi(i)\|\right\}^2 \tag{18}$$

Refer to Definition 1, the proof is completed. ■

As a special case of Theorem 1, when $d_i(k)=d(k)\in[d^m,d^M]$ in (9) - the time-varying delays of all subsystems are the same - Theorem 1 reduces to Corollary 1.

Corollary 1: System (9) with switching law (5) and $d_i(k)=d(k)\in[d^m,d^M]$ is mean square stable if there exist matrices $P>0, T>0, Q>0, S>0$ and $R>0$ with compatible dimensions such that

$$\begin{bmatrix}\bar{\Xi}_{11} & \bar{\Xi}_{12} & \bar{\Xi}_{13} & \bar{\Xi}_{14}\\ * & \bar{\Xi}_{22} & 0 & 0\\ * & * & \bar{\Xi}_{33} & 0\\ * & * & * & \bar{\Xi}_{44}\end{bmatrix} < 0 \tag{19}$$

where

$$
\begin{aligned}
\bar{\Xi}_{11} &= \begin{bmatrix} -P+T-S & S & 0 & 0 \\ * & -S-R+Q-T & R & 0 \\ * & * & -2R & R \\ * & * & * & -R-Q \end{bmatrix}, \\
\bar{\Xi}_{12} &= \begin{bmatrix} \bar{\Lambda}_1 & \cdots & \bar{\Lambda}_N \end{bmatrix}, \bar{\Xi}_{22} = diag\{-P, \cdots, -P\}, \\
\bar{\Lambda}_i &= \begin{bmatrix} \sqrt{\alpha_i} P A_i & 0 & \sqrt{\alpha_i} P A_{di} & 0 \end{bmatrix}^T \\
\bar{\Xi}_{13} &= \begin{bmatrix} d_1^m \bar{\Upsilon} S & \cdots & d_N^m \bar{\Upsilon} S \end{bmatrix}, \bar{\Xi}_{14} = \begin{bmatrix} d_1^r \bar{\Upsilon} R & \cdots & d_N^r \bar{\Upsilon} R \end{bmatrix}, \\
\bar{\Xi}_{33} &= diag\{-S, \cdots, -S\}, \bar{\Xi}_{44} = diag\{-R, \cdots, -R\}, \\
\bar{\Upsilon}_i &= \begin{bmatrix} \sqrt{\alpha_i} (A_i - I) & 0 & \sqrt{\alpha_i} A_{di} & 0 \end{bmatrix}^T.
\end{aligned}
$$

For completeness and comparisons, a sufficient condition for the mean square stability of an unforced switched system (1) with arbitrary switching and without any further information is given as Lemma 1 below.

Lemma 1: System (9) with $d_i(k) = d(k) \in [d^m, d^M]$ is mean square stable if there exist matrices $P_i > 0, T > 0, Q > 0, S > 0$ and $R > 0$ with compatible dimensions such that

$$
\begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} \\ * & -P_i & 0 & 0 \\ * & * & -S & 0 \\ * & * & * & -R \end{bmatrix} < 0 \tag{20}
$$

where Φ_{11} is obtained by replacing P in $\bar{\Xi}_{11}$ by P_i

$$
\begin{aligned}
\Phi_{12} &= \begin{bmatrix} P_i A_i & 0 & P_i A_{di} & 0 \end{bmatrix}^T, \\
\Phi_{13} &= \begin{bmatrix} d^m S (A_i - I) & 0 & d^m S A_{di} & 0 \end{bmatrix}^T, \\
\Phi_{14} &= \begin{bmatrix} d^r R (A_i - I) & 0 & d^r R A_{di} & 0 \end{bmatrix}^T.
\end{aligned}
$$

An unforced Arbitrary Jump System (AJS) is unstable if there is an unstable subsystem. For a randomly switched system with KSP, the system stability also depends on the sojourn probabilities. If the sojourn probabilities of stable subsystems are large, then it is more likely that the sufficient conditions set in Theorem 1 or Corollary 1 can be satisfied, leading to a conclusion of a stable system. This is demonstrated by Example 2 in Section 3, where one of the subsystems in this example is open-loop unstable.

C. Controller Design for Randomly Switched Systems with KSP

Theorem 2: System (7) with switching law (5) is mean square stable if there exist matrices $P > 0, T_i > 0, Q_i > 0, S_i > 0$, $R_i > 0$ and $K_i (i \in \Omega)$ with compatible dimensions such that

$$\begin{bmatrix} \Xi_{11} & \tilde{\Xi}_{12}P^{-1} & \tilde{\Xi}_{13} & \tilde{\Xi}_{14} \\ * & \tilde{\Xi}_{22} & 0 & 0 \\ * & * & \tilde{\Xi}_{33} & 0 \\ * & * & * & \tilde{\Xi}_{44} \end{bmatrix} < 0 \tag{21}$$

where Ξ_{11} is the same as that in (10), $\tilde{\Upsilon}$ and $\tilde{\Xi}_{12}$ are obtained from Υ and Ξ_{12} by replacing A_i with $A_i + B_i K_i$

$$\begin{aligned} \tilde{\Xi}_{22} &= diag\{-P^{-1}, -P^{-1}, \cdots, -P^{-1}\}, \\ \tilde{\Xi}_{13} &= \begin{bmatrix} d_1^m \tilde{\Upsilon} & \cdots & d_N^m \tilde{\Upsilon} \end{bmatrix}, \tilde{\Xi}_{14} = \begin{bmatrix} d_1^r \tilde{\Upsilon} & \cdots & d_N^r \tilde{\Upsilon} \end{bmatrix}, \\ \tilde{\Xi}_{33} &= diag\{-S_1^{-1}, \cdots, -S_1^{-1}, \cdots, -S_N^{-1}, \cdots, -S_N^{-1}\}, \\ \tilde{\Xi}_{44} &= diag\{-R_1^{-1}, \cdots, -R_1^{-1}, \cdots, -R_N^{-1}, \cdots, -R_N^{-1}\}. \end{aligned}$$

Proof: Replace A_i in (10) with $A_i + B_i K_i$ and define $\mathfrak{P}^{-1} = diag\{P^{-1}, \cdots, P^{-1}\}$, $\mathfrak{S}_i^{-1} = diag\{S_i^{-1}, \cdots, S_i^{-1}\}$, $\mathfrak{R}_i^{-1} = diag\{R_i^{-1}, \cdots, R_i^{-1}\}$, (21) can be obtained by pre and post-multiplying (10) with

$$diag\{I, I, I, I, \mathfrak{P}^{-1}, \mathfrak{S}_1^{-1}, \cdots, \mathfrak{S}_N^{-1}, \mathfrak{R}_1^{-1}, \cdots, \mathfrak{R}_N^{-1}\}.$$

■

Notice that the inequality in (21) is not a strict linear matrix inequality since the existence of P^{-1}, S_i^{-1} and R_i^{-1}. However, this form of inequality can be solved by using the cone complementarity linearization method [6].

The subsystem model is assumed linear in this paper. However, the KSP approach studied here can also be extended to study nonlinear switched systems, for example, systems with Randomly Occurring Nonlinearities (RONs) [4], [12], [13], [19].

A system with RONs can be modeled as

$$x(k+1) = Ax(k) + Bu(k) + \gamma(k) f(x(k)) \tag{22}$$

where $\gamma(k)$ is Bernoulli distributed white sequence taking values on 0 and 1. System (22) can be replaced by a randomly switched nonlinear systems with KSP

$$x(k+1) = \sum_{i=1}^{2} \gamma_i(k) \{A_i x(k) + B_i u(k) + f_i(x(k))\} \tag{23}$$

TABLE I

UPPER BOUNDS d^M OF EXAMPLE 1 FOR THE TWO CASES

α_1	0.99	0.9	0.7	0.5	0.3	0.1	0.01
α_2	0.01	0.1	0.3	0.5	0.7	0.9	0.99
Corollary 1	12	10	7	7	7	6	6
Lemma 1	6	6	6	6	6	6	6

by setting

$$\begin{aligned} A_i &= A, B_i = B, \gamma_1(k) = \gamma(k), \gamma_2(k) = 1 - \gamma(k), \\ f_1(x(k)) &= f(x(k)), f_2(x(k)) = 0. \end{aligned}$$

Therefore, switched system model (23) with KSP is a generalization of the systems with RONs in [4], [12], [13], [19].

Applying the KSP approach to nonlinear switched systems is a possible research direction in the future. Furthermore, another extension of the results in this paper is to study the stability of switched systems based on sojourn probabilities which are only partially known, or known with some uncertainties.

III. ILLUSTRATIVE EXAMPLES

Three numerical examples are shown in this section to demonstrate the application of the theoretical results developed in the last section.

Example 1: Consider an unforced randomly switched system of (9) with two subsystems, where

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.8 & 0 \\ 0 & 0.91 \end{bmatrix}, A_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.88 \end{bmatrix}, \\ A_{d1} &= \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, A_{d2} = \begin{bmatrix} 0.12 & 0 \\ 0.11 & 0.11 \end{bmatrix} \end{aligned}$$

Table I shows when $d^m = 1$ and for different sojourn probabilities, the upper bounds d^M of the delay $d(k)$ for two cases: (1) applying Corollary 1 with KSP, and (b) applying Lemma 1 assuming that sojourn probabilities are unknown and the system is treated as an AJS.

Next, set $\alpha_1 = 0.9$ and $\alpha_2 = 0.1$, the upper delay bounds d^M of the switched system when the lower bound d_m is set at different values (> 1) are shown in Table II for the two cases. This example demonstrates that less conservative results can be obtained when sojourn probabilities are known and are used for system stability analysis.

Example 2: Consider an unforced randomly switched system of (9) with two subsystems having the same time

TABLE II

UPPER BOUNDS d^M OF EXAMPLE 1 WITH $\alpha_1 = 0.9$, $\alpha_2 = 0.1$ AND DIFFERENT d^m

d^m	2	3	4	5	6
Corollary 1	10	10	10	11	11
Lemma 1	7	8	9	10	11

TABLE III

UPPER BOUNDS d^M OF EXAMPLE 2 WITH DIFFERENT KSP

α_1	0.9	0.8	0.7	0.6	0.5	0.4
α_2	0.1	0.2	0.3	0.4	0.5	0.6
	64	57	34	17	3	N/A

delay, and one subsystem with A_1 and A_{d1} below is stable but the other is unstable:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}, A_2 = \begin{bmatrix} 1.10 & 0 \\ 0 & 1.05 \end{bmatrix}, \\ A_{d1} &= A_{d2} = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.02 \end{bmatrix}, \end{aligned}$$

Clearly, if the system is an AJS, then it is unstable. However, if the sojourn probabilities are known, for certain sets of parameters, the system is stable. Table III shows the upper bounds d^M obtained when $d_m = 1$ for different KSPs. Notice that when $\alpha_1 = 0.4$ and $\alpha_2 = 0.6$, no feasible solutions can be found to satisfy Corollary 1. This is due to a large sojourn probability $(= 0.6)$ for the unstable subsystem.

Example 3: Consider the controller design for a randomly switched system (7) with two subsystems, where

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.9 & 0 \\ 0 & 1.1 \end{bmatrix}, A_2 = \begin{bmatrix} 1.13 & 0 \\ 0.16 & 0.47 \end{bmatrix}, \\ A_{d1} &= \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0.1 \end{bmatrix}, A_{d2} = \begin{bmatrix} 0.2 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \\ B_1 &= B_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, d_1^m = d_2^m = 1 \end{aligned}$$

For fixed $d_2^M = 3$, Table IV shows the upper bounds of d_1^M and controller feedback gain K_1 and K_2 obtained by applying Theorem 2.

For an initial conditions $\phi(k) = [0.3; 0.1]$, some simulation results and further information are presented in Figures 1-4.

Figure 1 and Figure 2 show the stable closed-loop time response of the system and the time-varying delays respectively when $\alpha_1 = 0.9, \alpha_2 = 0.1$. The state feedback matrices used in the simulation are given in the first column of Table IV. Within Figure 1, it also shows the switching sequence used in the simulation.

TABLE IV

THE RESULTS OBTAINED BY APPLYING THEOREM 2 FOR EXAMPLE 3

	$\alpha_1 = 0.9$ $\alpha_2 = 0.1$	$\alpha_1 = 0.7$ $\alpha_2 = 0.3$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$	$\alpha_1 = 0.3$ $\alpha_2 = 0.7$	$\alpha_1 = 0.1$ $\alpha_2 = 0.9$
d_1^M	3	3	3	4	5
K_1^T	$\begin{bmatrix} -0.3245 \\ -0.3468 \end{bmatrix}$	$\begin{bmatrix} -0.3404 \\ -0.3249 \end{bmatrix}$	$\begin{bmatrix} -0.3551 \\ -0.2969 \end{bmatrix}$	$\begin{bmatrix} -0.3259 \\ -0.2855 \end{bmatrix}$	$\begin{bmatrix} -0.3820 \\ -0.1391 \end{bmatrix}$
K_2^T	$\begin{bmatrix} -0.5402 \\ -0.0713 \end{bmatrix}$	$\begin{bmatrix} -0.5258 \\ -0.0290 \end{bmatrix}$	$\begin{bmatrix} -0.5349 \\ -0.0153 \end{bmatrix}$	$\begin{bmatrix} -0.5053 \\ -0.0184 \end{bmatrix}$	$\begin{bmatrix} -0.5984 \\ 0.0308 \end{bmatrix}$

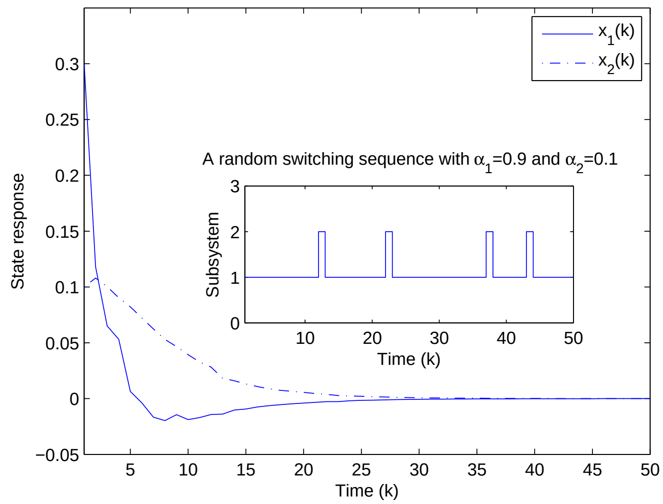

Fig. 1. The state responses for $\alpha_1 = 0.9$ and $\alpha_2 = 0.1$

Figure 3 and Figure 4 show the stable closed-loop time response of the system and the time-varying delays respectively when $\alpha_1 = 0.3, \alpha_2 = 0.7$. The state feedback matrices used in the simulation are given in the 4^{th} column of Table IV. Within Figure 3, it also shows the switching sequence used in the simulation.

IV. CONCLUSION

This paper studies stability analysis and stabilizing controller design for randomly switched systems with known sojourn probabilities (KSP) and two main theorems are developed for this purpose. Follow the KSP approach initiated in this paper, proposed further research includes (1) stability of a switched system with other types of subsystem dynamics, (2) stability of switched systems based on partially known sojourn probabilities or known with some uncertainties, (3) to study the links and to make comparisons between the KSP approach and the Markovian Jump System approach, (4) to study possible links between the KSP approach and dwell-time/average dwell-time based methods, and (5) possible extension of this approach to nonlinear switched systems.

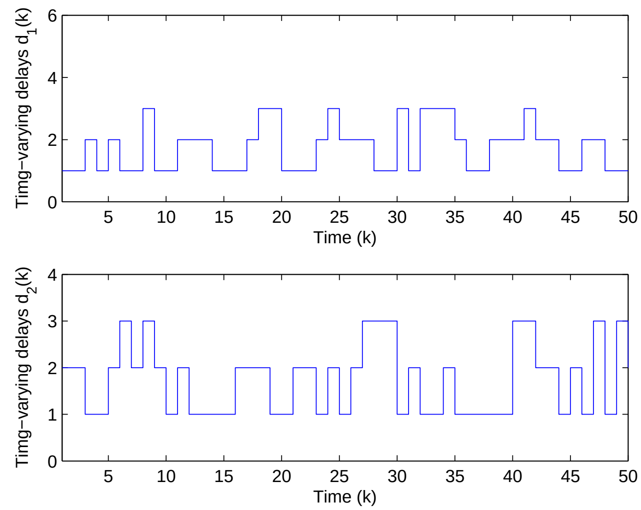

Fig. 2. Time-varying delays in the two subsystems

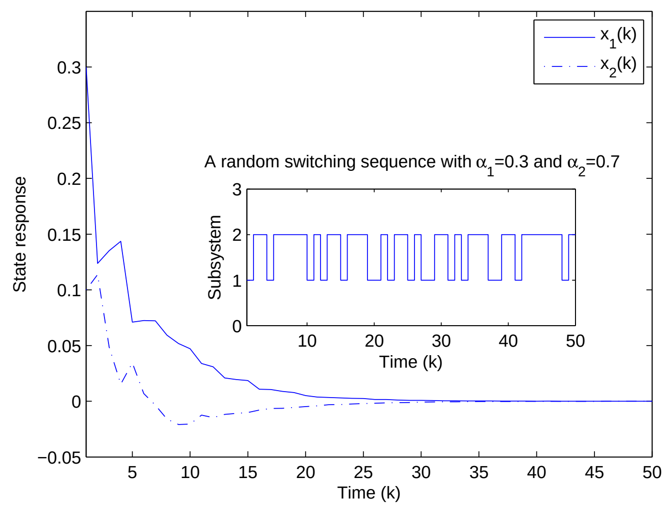

Fig. 3. The state responses for $\alpha_1 = 0.3$ and $\alpha_2 = 0.7$

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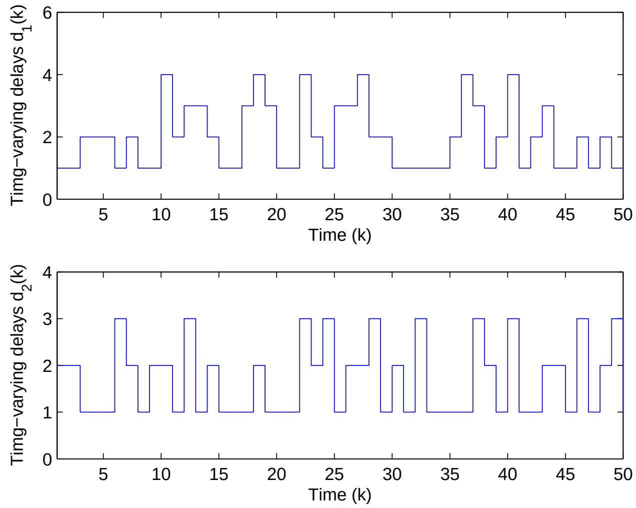

Fig. 4. Time-varying delays in the two subsystems

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